Def: An (infinite) sequence is an ordered list of numbers of the form

$$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$$

Ex 1 $0, 2, 4, 6, 8, 10, \ldots, ?, \ldots$

a_0 a_1 a_2 a_3 a_4 a_5 a_n

Notation: We can denote a sequence in the following ways:

(a) $\{a_n\}_{n=0}^{\infty}$

(b) $(a_n)_{n=0}^{\infty}$

(c) $a_n = f(n)$ for some function $f(x)$ – explicit form

(d) $a_{n+1} = f(a_n)$ for some function $f(x)$ – recursive form

Ex 2 Write the sequence from example 1 in explicit and recursive form.

Explicit: $a_n = 2n$ for $n \geq 0$

Recursive: $a_0 = 0$, $a_n = a_{n-1} + 2$ for $n \geq 1$

Def: For the sequence, a_n, the variable n is called the index variable

Note: We don't always use n for the index variable. Sometimes we use m, k, i, j instead.

Ex 3 Find an explicit formula for the sequence $\{7, 23, 39, 55, 71, 87, \dots\}$.

$a_0 = 7$, $a_{n+1} = a_n + 16$ for $n \geq 1$

Explicit: $a_n = 7 + 16n$ for $n \geq 0$

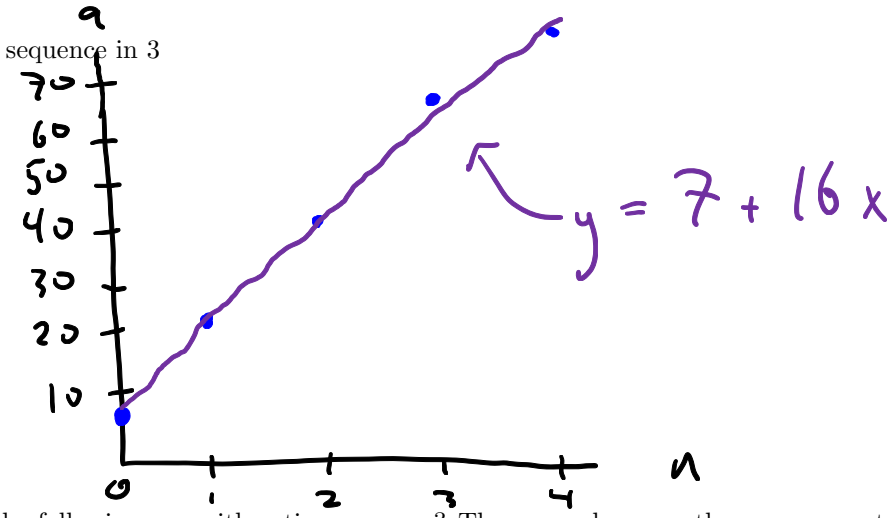
AIR-ith-METic

Def: An arithmetic sequence is a sequence where the difference between every pair of consecutive terms is constant.

Thm: Every arithmetic sequence has the form

$a_n = b + mn$ for constants b, m

Ex 4 Graph the sequence in 3



PQ. Which of the following are arithmetic sequences? There may be more than one correct answer.

(a) $a_n = 2n + 3$ for all $n \geqslant 0$

(b) $b_i = 2^i - 1$ for all $i \geqslant 0$

(c) $c_j = \ln(3j + 5)$ for all $j \geqslant 0$

(d) $d_k = -k$ for all $k \geqslant 0$

Ex 5 Find an explicit formula for the sequence defined by the recurrence relation $f_0 = 1$ and $f_{n+1} = \dfrac{f_n}{n+1}$ for all $n \geqslant 1$.

$f_0 = 1$

$f_1 = f_{0+1} = \dfrac{f_0}{0+1} = \dfrac{f_0}{1} = 1$

$f_2 = f_{1+1} = \dfrac{f_1}{1+1} = \dfrac{1}{2}$

$f_3 = \dfrac{f_2}{3} = \dfrac{1/2}{3} = \dfrac{1}{2 \cdot 3} = \dfrac{1}{6}$

$f_4 = \dfrac{f_3}{4} = \dfrac{1/2 \cdot 3}{4} = \dfrac{1}{2 \cdot 3 \cdot 4}$

$\vdots$

$f_n = \dfrac{1}{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n} = \dfrac{1}{n!}$

Ex 6 Find an explicit formula for the sequence defined by $x_0 = 3$ and $x_{n+1} = -5x_n$ for all $n \geqslant 1$.

$x_0 = 3$ 3

$x_1 = -5 \cdot x_0 = -5 \cdot 3$ -15

$x_2 = -5 \cdot x_1 = (-5) \cdot (-5) \cdot 3 = (-5)^2 \cdot 3$ 75

$x_3 = -5 \cdot x_2 = (-5) \cdot (-5)^2 \cdot 3 = (-5)^3 \cdot 3$ -375

$\vdots$

$x_n = (-5)^n \cdot 3$

Def: A geometric sequence is a sequence where the ratio between every pair of consecutive terms is constant.

Thm: Every geometric sequence has the form

$a_n = b \cdot r^n$ for constants b, r

PQ. Which of the following defines a geometric sequence? There may be more than one correct answer.

(a) $a_n = 2 \cdot 3^n$ for all $n \geqslant 0$

(b) $b_n = 2 + 3n$ for $n \geqslant 0$

(c) $c_n = 3\cos(\pi n)$ for all $n \geqslant 0$

(d) $d_n = \frac{1}{2n+7}$ for all $n \geqslant 0$

(e) $e_n = 4 \cdot n^3$ for all $n \geqslant 0$

(f) $f_n = -2 \cdot 4^{n+1}$ for all $n \geqslant 0$

Def: The Fibonacci sequence $(F_n)_{n=0}^{\infty}$ is defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}$$

Ex 7 Write out the first eight terms of the Fibonacci sequence and decide if $(F_n)_{n=0}^{\infty}$ is arithmetic, geometric, or neither.

n	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13

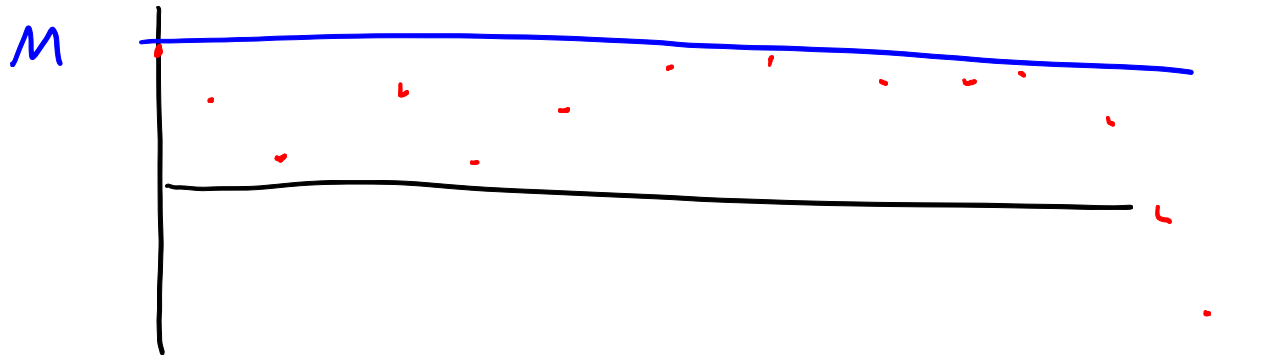
Diff: — 1 0 1 1 2 3 5

Ratio: — — 1 2 1.5 1.67 1.6 1.625

Neither

Def: Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence.

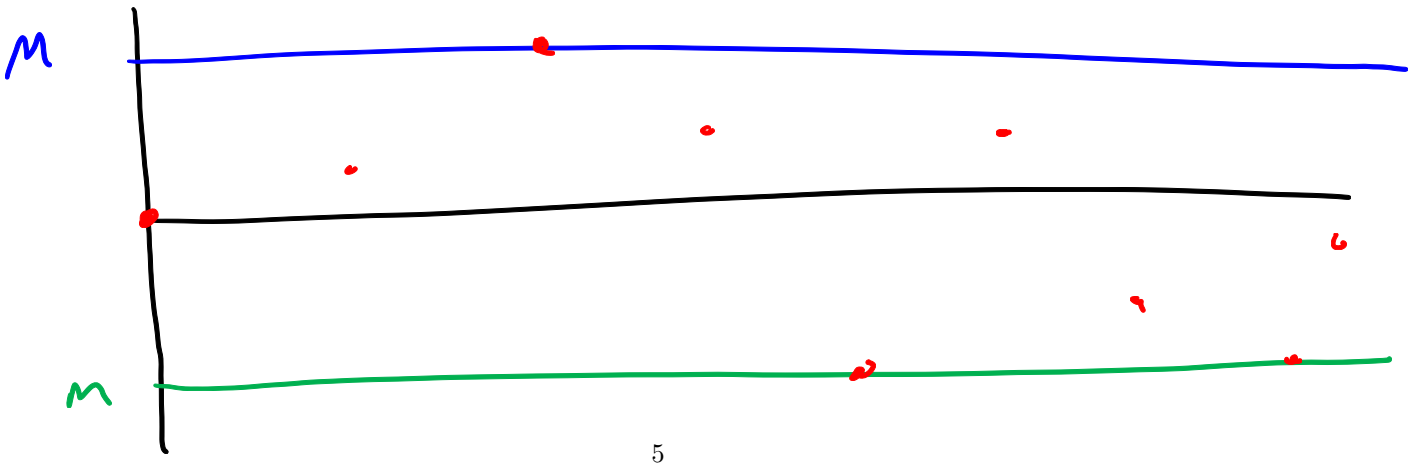
(a) $\{a_n\}_{n=0}^{\infty}$ is bounded from above if there exists a number M such that $a_n \leq M$ for all n.



(b) $\{a_n\}_{n=0}^{\infty}$ is bounded from below if there exists a number m such that $a_n \geq m$ for all n.



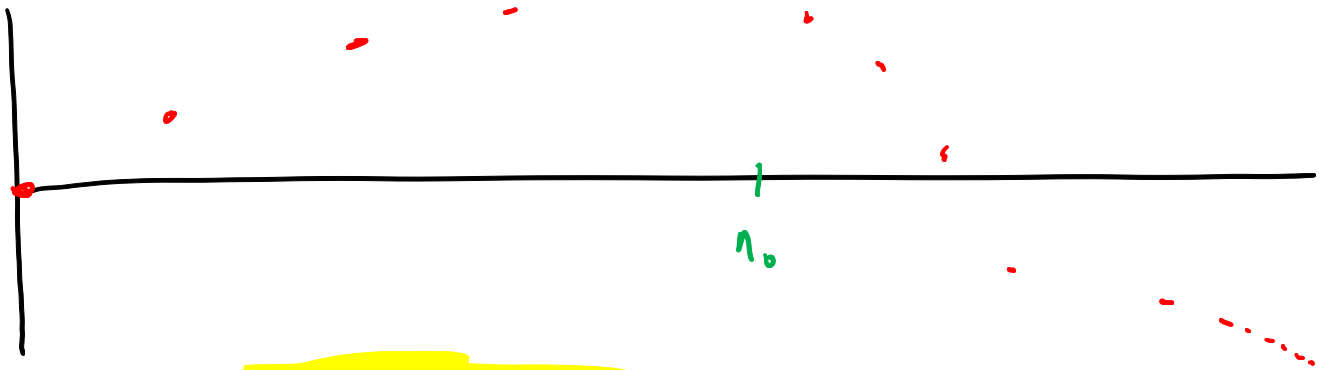
(c) $\{a_n\}_{n=0}^{\infty}$ is bounded if it is bounded from above and below



(d) $\{a_n\}_{n=0}^{\infty}$ is increasing after n_0 if $a_{n+1} \geq a_n$ for all $n \geqslant n_0$. We might also say that $\{a_n\}_{n=0}^{\infty}$ is eventually increasing



(e) $\{a_n\}_{n=0}^{\infty}$ is decreasing after n_0 if $a_{n+1} \leq a_n$ for all $n \geqslant n_0$. We might also say that $\{a_n\}_{n=0}^{\infty}$ is eventually decreasing



(f) $\{a_n\}_{n=0}^{\infty}$ is monotone after n_0 (or eventually monotone) if it is either increasing after n_0 or decreasing after n_0.

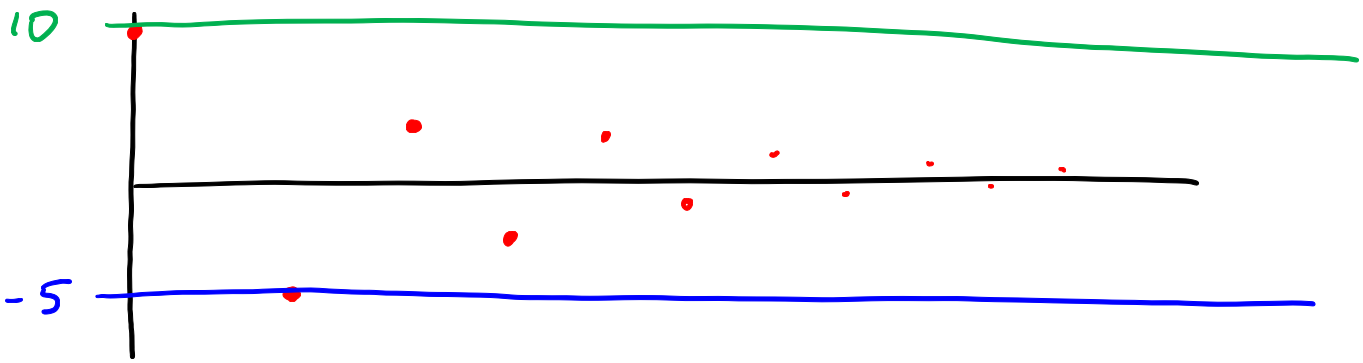
- either of the above pictures

Ex 8 Recall the Fibonacci Sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all $n \geqslant 1$. Is $\{F_n\}_{n=0}^{\infty}$ decreasing for all $n \geqslant 0$ or increasing for all $n \geqslant 0$ (or neither)? Is $\{F_n\}_{n=0}^{\infty}$ bounded below, above, or neither?

- bounded below by 0 (adding pos. #s together)
- increasing for $n \geq 0$ (same ↑)
- not decreasing for $n \geq 0$
- bounded above?
  - ↳ don't have tools, but it's plausible that it's not bounded from above

Ex 9 Consider the geometric sequence $\{y_n\}$ defined by $y_n = 10 \cdot \left(-\frac{1}{2}\right)^n$ for all $n \geqslant 1$. Is $\{y_n\}$ bounded? Is there an index n_0 such that $\{y_n\}$ is monotone for all $n \geqslant n_0$?

$y_0 = 10$, $y_1 = 10\left(-\frac{1}{2}\right) = -5$, $y_2 = \frac{5}{2}$, $y_3 = -\frac{5}{4}$, ...



Bounded from above by 10
Bounded from below by -5
→ bounded

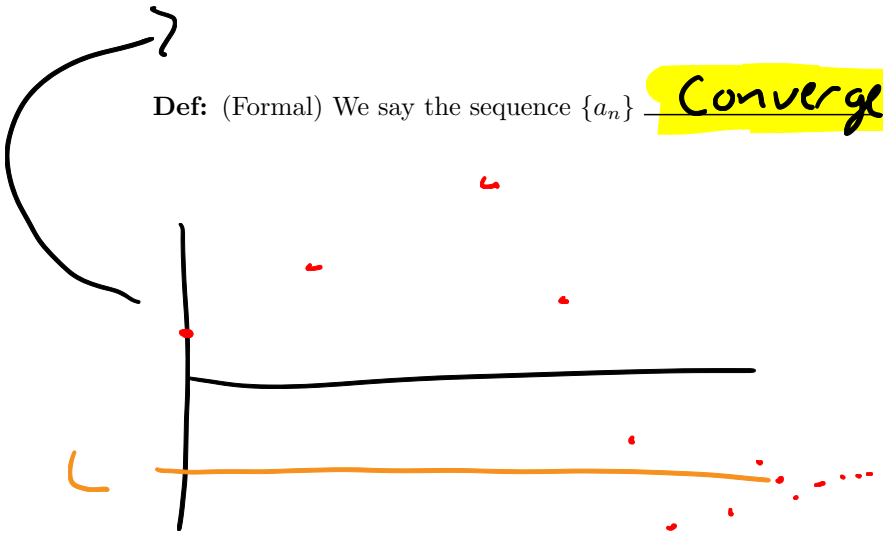
Never monotone (alternates inc → dec → inc → ...)

Def: (Informal) We say the sequence $\{a_n\}$ Converges to a number L if "a_n gets arbitrarily close to L" for "sufficiently large" values of n. In this case, we call $\{a_n\}$ a Convergent sequence and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \xrightarrow[n\to\infty]{} L \quad \text{or} \quad a_n \to L$$

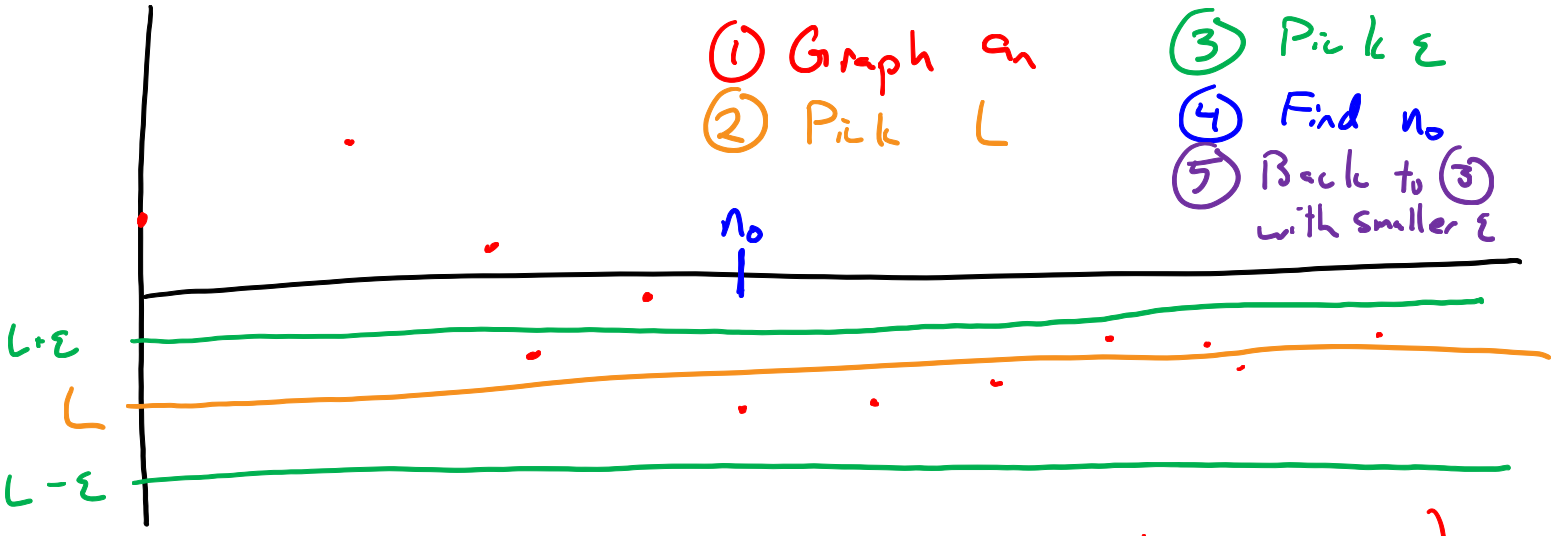
and we call L the limit of $\{a_n\}$.

Def: (Formal) We say the sequence $\{a_n\}$ Converges to the limit L if

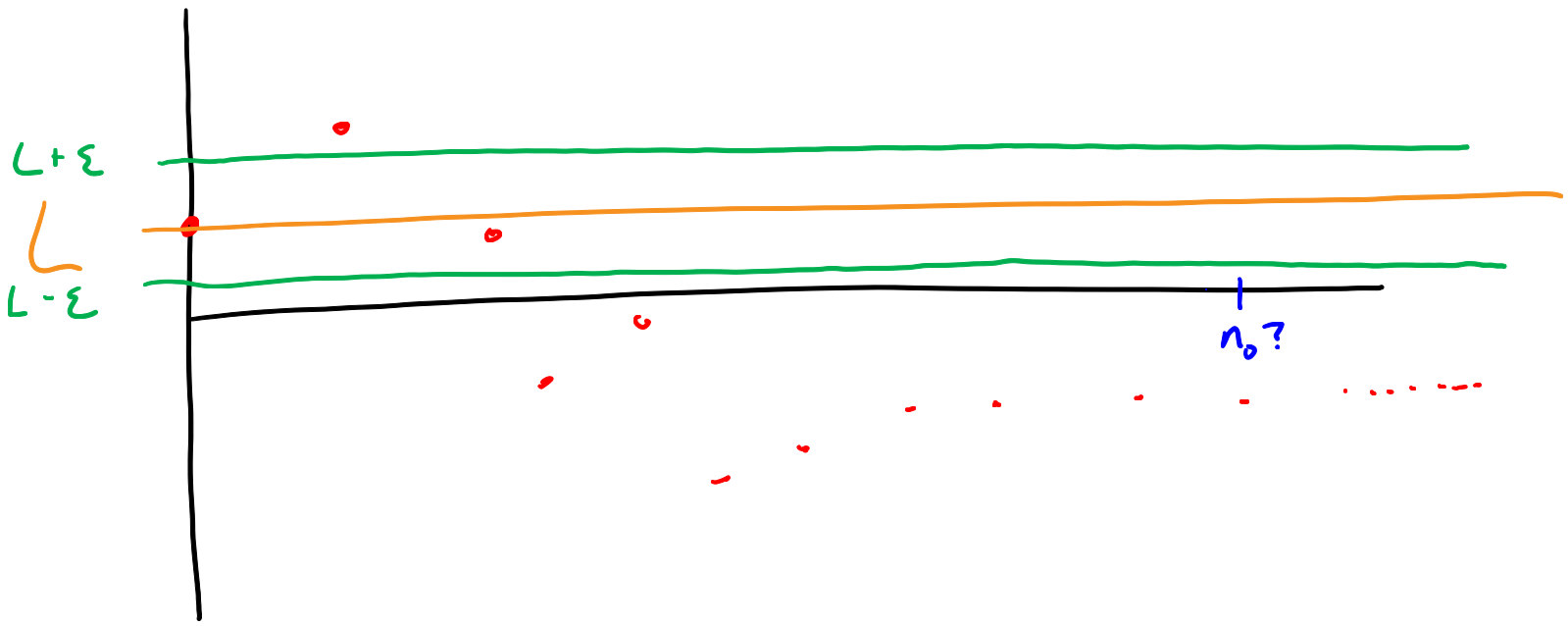


for every $\varepsilon > 0$ (epsilon), there exists n_0 so that

for all $n \geq n_0$, $|a_n - L| < \varepsilon$



* New page (conversation and divergent seq. pic.)

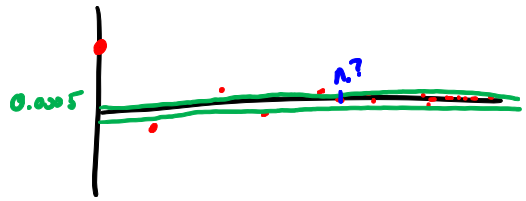


Conversation

- $\lim_{n\to\infty} a_n = L$ if we can have the following conversation

- You: When does a_n get within 1 of L?
- Me: For $n \geq 10$, $|a_n - L| < 1$ (n_0)
- You: When does a_n get within 0.1 of L?
- Me: For $n \geq 100$, $|a_n - L| < 0.1$ (n_0)
- You: When does a_n get within 0.01 of L?
- Me: For $n \geq 1000$, $|a_n - L| < 0.001$ (n_0)

Ex 10 Suppose $x_n = \left(-\frac{1}{7}\right)^n$ for all $n \geqslant 1$. Let $\varepsilon = 0.0005$. Find an explicit integer N with the property that $|x_n - 0| < \varepsilon$ for all $n \geqslant N$.



Want: n_0 s.t. $\left|\left(-\frac{1}{7}\right)^{n_0}\right| < 0.0005$

LHS: $\left|\left(-\frac{1}{7}\right)^{n_0}\right| = \left(\frac{1}{7}\right)^{n_0} = \frac{1}{7^{n_0}}$

Want: $\frac{1}{7^{n_0}} < 0.0005 \longleftrightarrow \frac{1}{0.0005} < 7^{n_0}$

$\longleftrightarrow \log_7\left(\frac{1}{0.0005}\right) < n_0$

$\$\$$

3.9

$n_0 = 4$

Ex 11 How do you convince someone $\lim_{n\to\infty} \frac{2n^2}{1+n^2} = 2$?

Let $\varepsilon > 0$.

Want: n_0 s.t. $\left|\frac{2n_0^2}{1+n_0^2} - 2\right| < \varepsilon$

LHS: $\left|\frac{2n_0^2}{1+n_0^2} - 2\right| = 2 - \frac{2n_0^2}{1+n_0^2} = \frac{2(1+n_0^2) - 2n_0^2}{1+n_0^2}$

$= \frac{2}{1+n_0^2}$

Want: n_0 s.t. $\frac{2}{1+n_0^2} < \varepsilon \longleftrightarrow \frac{2}{\varepsilon} < 1 + n_0^2$

$\longleftrightarrow \frac{2}{\varepsilon} - 1 < n_0^2$

$\longleftrightarrow \sqrt{\frac{2}{\varepsilon} - 1} < n_0$

If $n \geqslant \sqrt{\frac{2}{\varepsilon} - 1}$, then $\left|\frac{2n^2}{1+n^2} - 2\right| < \varepsilon$

Ex 12 Compute $\lim_{n\to\infty} r^n$ for any (constant) r.

Ask for values of $r \neq 0$ that give diff. answers

$$\lim_{n\to\infty} r^n \begin{cases} \text{DNE} & \text{if } r \leq -1 \\ = 0 & \text{if } -1 < r < 1 \\ = 1 & \text{if } r = 1 \\ = \infty & \text{if } r > 1 \end{cases}$$

Def: If a sequence $\{a_n\}$ does not converge to any limit, we say $\{a_n\}$ diverges or is divergent

Note A sequence can diverge in the following ways:

(a) $\lim_{n\to\infty} a_n = \infty$

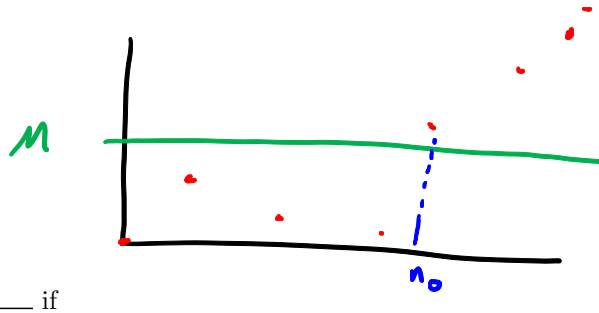
(b) $\lim_{n\to\infty} a_n = -\infty$

(c) $\lim_{n\to\infty} a_n$ DNE

Def: A sequence $\{a_n\}$ diverges to ∞ if

for every $M > 0$, there exists n_0 so that for all $n > n_0$, $a_n > M$.

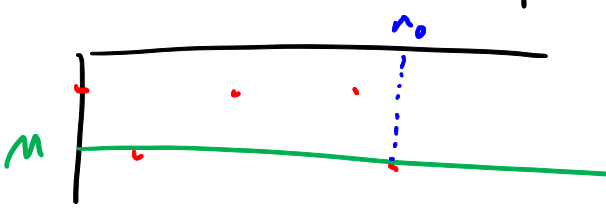
* pic



Def: A sequence $\{a_n\}$ diverges to $-\infty$ if

for every $M < 0$, there exists n_0 so that for all $n \leq n_0$, $a_n < M$

* pic



Thm: (Squeeze Theorem for Sequences) Suppose $a_n \leq b_n \leq c_n$ for all $n \geqslant n_0$ and suppose that $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$. Then $\lim_{n\to\infty} b_n = L$

* pic

Ex 13 Let $b_n = \dfrac{\sin(2^n)}{2^n}$. Compute $\lim_{n\to\infty} b_n$.

$$-1 \leq \sin(2^n) \leq 1$$

$$\to \quad \frac{-1}{2^n} \leq \frac{\sin(2^n)}{2^n} \leq \frac{1}{2^n}$$

$$\to \quad \lim_{n\to\infty} -\frac{1}{2^n} \leq \lim_{n\to\infty} \frac{\sin(2^n)}{2^n} \leq \lim_{n\to\infty} \frac{1}{2^n}$$

$$\text{"0"} \qquad\qquad \text{"0"} \qquad\qquad \text{"0"}$$

* new page

Ex 14 Compute $\lim_{n\to\infty} \dfrac{n!}{n^n}$. But guess first!

PQ.
(a) 0
(b) 1
(c) ∞
(d) $-\infty$
(e) e
(f) π
(g) The limit does not exist

Thm: (Algebraic Limit Laws) If $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$, and c is a constant, then:

(a) $\lim_{n\to\infty} c = c$

(b) $\lim_{n\to\infty} ca_n = c \cdot \lim_{n\to\infty} a_n = cA$

(c) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$

(d) $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = A \cdot B$

(e) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $b_n \neq 0$ for all n and $B \neq 0$.

Ex 15 Determine whether the sequence $\left\{\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$ converges. If it converges, then compute the limit.

$\lim_{n\to\infty} \frac{3n^4 - 7n^2 + 5}{6 - 4n^4} \stackrel{?}{=} \frac{\infty}{\infty}$

$\frac{3n^4 - 7n^2 + 5}{6 - 4n^4} = \frac{n^4\left(3 - \frac{7}{n^2} + \frac{5}{n^4}\right)}{n^4\left(\frac{6}{n^4} - 4\right)} = \frac{3 - \frac{7}{n^2} + \frac{5}{n^4}}{\frac{6}{n^4} - 4}$

$\lim_{n\to\infty} \frac{3 - \frac{7}{n^2} + \frac{5}{n^4}}{\frac{6}{n^4} - 4} = \frac{\lim_{n\to\infty} 3 - \frac{7}{n^2} + \frac{5}{n^4}}{\lim_{n\to\infty} \frac{6}{n^4} - 4} = -\frac{3}{4}$

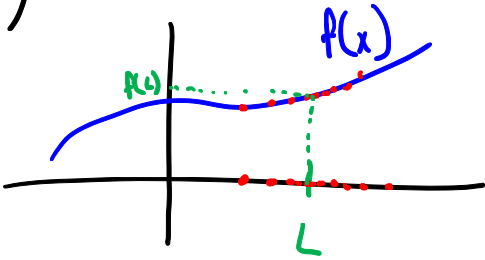
bk denom $\to$ # (not 0)

Ex 16 Why can't we use the Algebraic Limit Laws to directly compute $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$?

$\sqrt{n+1}$ diverges

Thm: (Continuous Functions Defined on Convergent Sequences) Suppose that $\{a_n\}$ and L are in the domain of a function $f(x)$. Suppose that $f(x)$ is continuous at L and that $a_n \to L$ ($n\to\infty$). Then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$



* pic

Ex 17 Compute $\lim_{n\to\infty}(\ln(2+3n)-\ln(n))$.

$$= \lim_{n\to\infty} \ln\left(\frac{2+3n}{n}\right)$$

$$= \lim_{n\to\infty} \ln\left(\frac{2}{n}+3\right)$$

Recall: $\ln(x)$ is cts. for all $x>0$

$\frac{2}{n}+3 \xrightarrow{n\to\infty} 3$ (positive)

$$\Rightarrow \lim_{n\to\infty} \ln\left(\frac{2}{n}+3\right) = \ln\left(\lim_{n\to\infty}\frac{2}{n}+3\right) = \ln(3)$$

Q. How are boundedness and monotonicity related to convergence?

Thm: (Monotone Convergence) If a sequence $\{a_n\}$ is bounded and monotone for all $n \geqslant n_0$, then a_n converges to some limit, L.

Ex 18 $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} = ?$

$g_0 = 0$, $g_{n+1} = \sqrt{1+g_n}$

$\rightarrow g_0 = 0$, $g_1 = \sqrt{1+0} = \sqrt{1}$, $g_2 = \sqrt{1+\sqrt{1}}$, $g_3 = \sqrt{1+\sqrt{1+\sqrt{1}}}$...

(1) g_n is monotone for $n \geq 0$

$g_0 \leq g_1$

$1+g_0 \leq 1+g_1$

$\sqrt{1+g_0} \leq \sqrt{1+g_1}$

$g_1 \leq g_2$

$1+g_1 \leq 1+g_2$

$\sqrt{1+g_1} \leq \sqrt{1+g_2}$

$g_2 \leq g_3$

$\vdots$

$g_0 \leq g_1 \leq g_2 \leq \cdots$

(2) g_n is bounded: $0 \le g_n \le 2$ for all n

↳ $0 \le g_0 \le 2$ ✓

$1 \le g_0 + 1 \le 3$

$\sqrt{1} \le \sqrt{g_0 + 1} \le \sqrt{3}$

$1 \le g_1 \le \sqrt{3}$

$0 \le g_1 \le 2$

$1 \le g_1 + 1 \le 3$

$\sqrt{1} \le \sqrt{g_1 + 1} \le \sqrt{3}$

$1 \le g_2 \le \sqrt{3}$

$0 \le g_2 \le 2$

$\vdots$

By MCT, g_n converges, say $\lim_{n \to \infty} g_n = L$

$g_{n+1} = \sqrt{1 + g_n} \rightarrow g_{n+1}^2 = 1 + g_n$

$\rightarrow \lim_{n \to \infty} g_{n+1}^2 = 1 + \lim_{n \to \infty} g_n$

$\rightarrow L^2 = 1 + L$

$\rightarrow L = \frac{1 + \sqrt{5}}{2}$